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The Berezin inequality on domains of infinite measure

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Abstract The Berezin inequality gives an upper bound on the Riesz means of the magnetic Schrödinger operator on a set of finite volume. We find an analogous inequality for the magnetic operator with homogeneous magnetic field on sets whose complement in $\mathbb{R}^2$ has finite measure. Similar bounds are obtained for the Heisenberg sub-Laplacian.

Keywords Berezin inequality · Magnetic field · Heisenberg sub-Laplacian

Mathematics Subject Classification (2000) 35J10 · 35P15 · 35P20

1 Introduction

Let Ω be an open subset of $\mathbb{R}^d$ such that $\Omega^{\mathsf{c}} = \mathbb{R}^d \setminus \Omega$ is of finite measure. In [2] Berezin proved that the Dirichlet Laplacian operator $-\Delta_{\Omega^{\mathsf{c}}}^{\mathscr{D}}$ on Ω^{c} satisfies the inequality

$$\operatorname{tr}(-\Delta_{\Omega^{\mathsf{c}}}^{\mathscr{D}} - \lambda)_-^{\gamma} \leq (2\pi)^{-d}|\Omega^c| \int\limits_{\mathbb{R}^d} (|p|^2 - \lambda)_-^{\gamma}\, \mathrm{d}p = L_{\gamma,d}^{\mathrm{cl}}|\Omega^c|\lambda^{\gamma+\frac{d}{2}} \tag{1}$$

for all $\lambda \geq 0, \gamma \geq 1$. Here and below, the measure of a set $S \subset \mathbb{R}^d$ is denoted by $|S|$ and $x_- = \frac{1}{2}(|x| - x)$ is the negative part of a variable, a function or a self-adjoint operator. The so-called Lieb–Thirring constant $L_{\gamma,d}^{\mathrm{cl}}$ can be computed to be

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$$L_{\gamma,d}^{\mathrm{cl}} = (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\gamma+1)}{\Gamma\left(\gamma+1+\frac{d}{2}\right)}$$

and is sharp, which follows from an asymptotic result by Weyl [17]. For the inequality to hold it is essential that the Laplace operator is considered on the set Ω^{c} of finite volume. This guarantees that H only has discrete spectrum consisting of eigenvalues converging to infinity, showing that the left-hand-side of (1) exists.

In [5] a similar result to the Berezin inequality (1) has been established for the Dirichlet Laplace operator on the set Ω of infinite measure. To this end one introduces the orthogonal projection $P_\Omega : L^2(\mathbb{R}^d) \to L^2(\Omega)$, i.e. the multiplication with the characteristic function χ_Ω. The operator $P_\Omega(-\Delta)P_\Omega$ corresponds to the Laplacian on the set Ω with Dirichlet boundary conditions. Since the continuous spectrum of $P_\Omega(-\Delta)P_\Omega$ contains the positive real axis, the operator $(P_\Omega(-\Delta)P_\Omega - \lambda)_-$ is not trace-class on $L^2(\mathbb{R}^d)$. However, it can be compared to a suitable operator to achieve similar results to (1). The authors of [5] considered the difference $(-\Delta - \lambda)_- - (P_\Omega(-\Delta)P_\Omega - \lambda)_-$ and proved that

$$\operatorname{tr}\left((-\Delta-\lambda)_- - (P_\Omega(-\Delta)P_\Omega - \lambda)_-\right) \geq L_{1,d}^{\mathrm{cl}} |\mathbb{R}^d \setminus \Omega| \lambda^{1+\frac{d}{2}}, \tag{2}$$

which can be seen as an analogue of the Berezin inequality for perturbations of the continuous spectrum of the Laplace operator. Bounds on traces for these types of problems are a fairly recent research area and we point to [5] for a generalisation of Lieb–Thirring inequalities to this setting.

In our paper we aim to find an analogous inequality to (2) for the magnetic operator $H_B = (-\mathrm{i}\nabla + A(x))^2$. Similar to the case of the Laplacian, problems stem from the fact that $(P_\Omega H_B P_\Omega - \lambda)_-$ is not trace-class. Thus we consider the difference $(H_B-\lambda)_-^\gamma - (P_\Omega H_B P_\Omega - \lambda)_-^\gamma$ and establish lower bounds on the trace of this operator. We also prove a similar inequality for the sub-Laplacian L on the first Heisenberg group $\mathbb{H}^1$.A key observation for our results is that, for any self-adjoint operator H, a formal computation involving the Berezin–Lieb inequality for convex functions (see [1] and [14]) yields the result

$$\operatorname{tr}\left((H-\lambda)_-^\gamma - (P_\Omega H P_\Omega - \lambda)_-^\gamma\right) \geq \operatorname{tr}\left((H-\lambda)_-^\gamma - P_\Omega (H-\lambda)_-^\gamma P_\Omega\right). \tag{3}$$

It is the object of this work to give correct mathematical meaning to this observation and to explicitly calculate the right-hand-side for the two special choices of H.

The Berezin inequality (1) on domains of finite measure has inspired a number of authors and is related to the Li–Yau inequality [13]. In their paper the authors showed that the sum over the first k eigenvalues $\lambda_1, \ldots, \lambda_k$ of $-\Delta_{\Omega^{\mathsf{c}}}^{\mathscr{D}}$ can be bounded from below as

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \left(L_{0,d}^{\mathrm{cl}} |\Omega^{\mathsf{c}}|\right)^{-\frac{2}{d}} k^{\frac{d+2}{d}}.$$

This was later proven to be a corollary of (1) via the Legendre transformation, see [11]. In [9] comparable inequalities were established for various classes of differential and pseudo-differential operators including $((-\Delta)^{\alpha})^{\mathcal{D}}_{\Omega^{\mathsf{c}}}$ with $\alpha > 0$. A similar inequality to (1) can be found for Schrödinger operators with magnetic fields in the case $d = 2$. The operator $H^{\mathscr{D}}_{B,\Omega^{\mathsf{c}}} := (-\mathrm{i}\nabla + A(x))^2$ on $L^2(\Omega^{\mathsf{c}})$ with Dirichlet boundary conditions and arbitrary vector field A satisfies

$$\operatorname{tr}(H^{\mathscr{D}}_{B,\Omega^{\mathsf{c}}} - \lambda)^{\gamma}_{-} \leq L^{\mathrm{cl}}_{\gamma,2}\lambda^{\gamma+1}|\Omega^{\mathsf{c}}| \tag{4}$$

for all $\gamma \geq \frac{3}{2}$, which follows from a result by Laptev and Weidl in [12] (see also [6]). In [4] this was generalised to $\gamma \geq 1$ under the restriction that the magnetic field $B = \mathrm{d}A$ is constant. In this case the upper bound in (4) can be improved by allowing it to depend on B

$$\operatorname{tr}(H^{\mathscr{D}}_{B,\Omega^{\mathsf{c}}} - \lambda)^{\gamma}_{-} \leq |\Omega^{\mathsf{c}}|\frac{B}{2\pi}\sum_{k=0}^{\infty}((2k+1)B - \lambda)^{\gamma}_{-} \tag{5}$$

as shown in [6]. In their paper the authors also proved that, under the assumption that Ω^{c} is a tiling domain, this inequality also holds if $0 \leq \gamma < 1$, where it is sharp. For $\gamma = 1$ the right-hand-side of (5) can be adapted to magnetic operators with additional external potentials V, see [7]. The Berezin inequality was furthermore extended to the sub-Laplacian L on the Heisenberg group $\mathbb{H}^1$. In [8] (see also [16]), it was proven that the Dirichlet realisation $L^{\mathscr{D}}_{\Omega^{\mathsf{c}}}$ of L on a domain $\Omega^{\mathsf{c}} \subset \mathbb{H}^1$ of finite measure satisfies

$$\operatorname{tr}(L^{\mathscr{D}}_{\Omega^{\mathsf{c}}} - \lambda)^{\gamma}_{-} \leq |\Omega^{\mathsf{c}}|\frac{1}{16}\frac{1}{(\gamma+1)(\gamma+2)}\lambda^{\gamma+2}. \tag{6}$$

In our paper we obtain lower bounds on the traces of the differences $(H_B - \lambda)^{\gamma}_{-} - (P_{\Omega}H_B P_{\Omega} - \lambda)^{\gamma}_{-}$ and $(L - \lambda)^{\gamma}_{-} - (P_{\Omega}LP_{\Omega} - \lambda)^{\gamma}_{-}$ which are of the same form as the upper bounds in (5) and (6), respectively.

The paper is organised as follows. In Sect. 2 we discuss (3) in the general setting of H being a self-adjoint operator on $L^2(\mathbb{R}^d)$. We then state our main results for the magnetic operator H_B with constant magnetic field and the sub-Laplacian L in Theorems 2 and 3, respectively. The complete proofs of these results are given in the subsequent sections.

2 Statement of the main results

Let H be a self-adjoint operator on the space $L^2(\mathbb{R}^d)$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function such that $\varphi(H) - \varphi(P_{\Omega}HP_{\Omega})$ and $\varphi(H) - P_{\Omega}\varphi(H)P_{\Omega}$ are both trace-class. Under these assumptions a generalisation of the Berezin–Lieb inequality as derived in [10] states that

$$\operatorname{tr}\,(P_{\Omega}\varphi(H)P_{\Omega} - \varphi(P_{\Omega}HP_{\Omega})) \geq 0.$$

As a consequence, we obtain the inequality

$$\operatorname{tr}\left(\varphi(H)-\varphi(P_\Omega H P_\Omega)\right) \geq \operatorname{tr}\left(\varphi(H)-P_\Omega\varphi(H)P_\Omega\right) \tag{7}$$

by making use of the additivity of the trace. We now simplify the right-hand-side of (7) as follows. To shorten notation denote the trace-class operator $Q := \varphi(H) - P_\Omega\varphi(H)P_\Omega$ and let $P_{\Omega^c} = \mathbb{I} - P_\Omega$ be the complementary projection of P_Ω. Clearly Q can be written as the sum of four operators corresponding to the decomposition of $L^2(\mathbb{R}^d)$ into $\operatorname{ran} P_\Omega$ and $\operatorname{ran} P_{\Omega^c}$, i.e.

$$Q = P_\Omega Q P_\Omega + P_\Omega Q P_{\Omega^c} + P_{\Omega^c} Q P_\Omega + P_{\Omega^c} Q P_{\Omega^c}.$$

In [15][Theorem VI.25] it is shown that, if T is trace-class and S is bounded, then $\operatorname{tr}(ST) = \operatorname{tr}(TS)$. As a result $\operatorname{tr}(P_\Omega Q P_{\Omega^c}) = 0$ as well as $\operatorname{tr}(P_{\Omega^c} Q P_\Omega) = 0$. Thus the trace of Q consists only of the diagonal terms

$$\operatorname{tr}\left(\varphi(H) - P_\Omega\varphi(H)P_\Omega\right) = \operatorname{tr}(P_\Omega Q P_\Omega + P_{\Omega^c} Q P_{\Omega^c}) = \operatorname{tr}\left(P_{\Omega^c}\varphi(H)P_{\Omega^c}\right).$$

These results are summarised in the following theorem.

Theorem 1 *Let H be a self-adjoint operator on $L^2(\mathbb{R}^d)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function such that $\varphi(H) - \varphi(P_\Omega H P_\Omega)$ and $\varphi(H) - P_\Omega\varphi(H)P_\Omega$ are both trace-class. Then the Berezin–Lieb type inequality*

$$\operatorname{tr}\left(\varphi(H)-\varphi(P_\Omega H P_\Omega)\right) \geq \operatorname{tr}\left(\varphi(H)-P_\Omega\varphi(H)P_\Omega\right) = \operatorname{tr}\left(P_{\Omega^c}\varphi(H)P_{\Omega^c}\right)$$

holds.

While this result is true for arbitrary self-adjoint operators H, we shall now apply it to two special choices of H to obtain the main results of this work. First, consider Schrödinger operators with magnetic fields. Let the magnetic field $B(x)$ be a two-form on $\mathbb{R}^d$ and the magnetic vector potential $A(x)$ a one-form satisfying $B(x) = \mathrm{d}A(x)$. We shall restrict ourselves to the case $d = 2$ and in the remainder of this work, we furthermore assume that B is constant and positive. Consider the magnetic operator $H_B = (-\mathrm{i}\nabla + A(x))^2$, which is defined as the closure of the form

$$(\psi, H_B\psi) := \int_{\mathbb{R}^2} \left|(-\mathrm{i}\nabla + A(x))\,\psi(x)\right|^2 \,\mathrm{d}x$$

on $\mathscr{C}_c^\infty(\mathbb{R}^2)$, the set of smooth functions with compact support. The obtained operator is found to be self-adjoint and we can state the first main result.

Theorem 2 *Assume $d = 2$, $\lambda \geq 0$, $B > 0$, $\gamma \geq 1$ and let Ω be an open subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \Omega$ has finite measure. Then the inequality*

$$\begin{aligned}\operatorname{tr}\left((H_B-\lambda)_-^\gamma - (P_\Omega H_B P_\Omega - \lambda)_-^\gamma\right) &\geq \operatorname{tr}\left((H_B-\lambda)_-^\gamma - P_\Omega (H_B-\lambda)_-^\gamma P_\Omega\right)\\ &= \operatorname{tr}\left(P_{\Omega^c}(H_B-\lambda)_-^\gamma P_{\Omega^c}\right)\end{aligned} \tag{8}$$

holds and the right-hand-side can be calculated explicitly as

$$\operatorname{tr}\left(P_{\Omega^c}(H_B-\lambda)_-^\gamma P_{\Omega^c}\right)=|\mathbb{R}^2\setminus\Omega|\frac{B}{2\pi}\sum_{k=0}^{\infty}((2k+1)B-\lambda)_-^\gamma\,. \tag{9}$$

The proof of Theorem 2 is provided in Sect. 3. The lower bound (9) coincides with the upper bound (5) for the magnetic operator on the set Ω^c of finite volume. In essence the proof is the same.

Similar results can also be obtained on the first Heisenberg group $\mathbb{H}^1$. Here, $\mathbb{H}^1$ is considered to be the three-dimensional space $\mathbb{R}^3$ equipped with the non-commutative multiplication

$$(x_1,x_2,x_3)\circ(y_1,y_2,y_3)=\left(x_1+y_1,x_2+y_2,x_3+y_3-\frac{1}{2}(x_1y_2-x_2y_1)\right)$$

for $(x_1,x_2,x_3),(y_1,y_2,y_3)\in\mathbb{R}^3$. On the Heisenberg group, we introduce the two left-invariant vector fields

$$X_1=\frac{\partial}{\partial x_1}+\frac{1}{2}x_2\frac{\partial}{\partial x_3},\quad X_2=\frac{\partial}{\partial x_2}-\frac{1}{2}x_1\frac{\partial}{\partial x_3}.$$

Using these definitions, we consider the quadratic form

$$\ell(\psi)=\int_{\mathbb{R}^3}\left(|X_1\psi|^2+|X_2\psi|^2\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

on $\mathcal{C}_c^\infty(\mathbb{R}^3)$ and note that the closure of this form gives the self-adjoint sub-Laplacian $L=-X_1^2-X_2^2$ on $\mathbb{H}^1$. For a detailed background we refer to the literature, e.g. [3]. The sub-Laplacian L is found to satisfy the following analogue of the Berezin inequality.

Theorem 3 *Assume $\lambda\geq 0$, $\gamma\geq 1$ and let Ω be an open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3\setminus\Omega$ has finite measure. Then the inequality*

$$\begin{aligned}\operatorname{tr}\left((L-\lambda)_-^\gamma-(P_\Omega LP_\Omega-\lambda)_-^\gamma\right)&\geq\operatorname{tr}\left((L-\lambda)_-^\gamma-P_\Omega(L-\lambda)_-^\gamma P_\Omega\right)\\&=\operatorname{tr}\left(P_{\Omega^c}(L-\lambda)_-^\gamma P_{\Omega^c}\right)\end{aligned} \tag{10}$$

holds for the sub-Laplacian L on $\mathbb{H}^1$ and the right-hand-side can be calculated explicitly as

$$\operatorname{tr}\left(P_{\Omega^c}(L-\lambda)_-^\gamma P_{\Omega^c}\right)=|\mathbb{R}^3\setminus\Omega|\frac{1}{16}\frac{1}{(\gamma+1)(\gamma+2)}\lambda^{\gamma+2}. \tag{11}$$

Similarly to the previous application, the lower bound (11) coincides with the upper bound in the case of the Heisenberg sub-Laplacian being defined on the domain Ω^c

of finite measure with Dirichlet boundary conditions, see (6). The proof of Theorem 3 is basically the same as in the case of finite measure [8] and can be found in Sect. 4. Note that this result can easily be generalised to the N-th Heisenberg group $\mathbb{H}^N$.

Remark 1 Using Theorem 1 we can also reproduce the results of Frank, Lewin, Lieb and Seiringer [5] and show that

$$\operatorname{tr}\left((-\Delta-\lambda)_-^\gamma - (P_\Omega(-\Delta)P_\Omega-\lambda)_-^\gamma\right) \geq L_{\gamma,d}^{\mathrm{cl}} |\mathbb{R}^d \setminus \Omega| \lambda^{\gamma+\frac{d}{2}}$$

for the Laplacian on a set $\Omega \subset \mathbb{R}^d$ with complement of finite measure.

3 The proof of Theorem 2

Let $\varphi_{\lambda,\gamma} : \mathbb{R} \to \mathbb{R}$ be the convex function defined as

$$\varphi_{\lambda,\gamma}(t) = (t-\lambda)_-^\gamma = \begin{cases} (\lambda-t)^\gamma, & t \leq \lambda \\ 0, & t > \lambda. \end{cases} \tag{12}$$

Applying Theorem 1 to this function and the operator H_B yields (8) and it only remains to prove (9). This can be done in complete analogy to calculations by Frank, Loss and Weidl [6]. The spectrum of H_B is entirely discrete and can be calculated to be $(2k+1)B$ for $k \in \mathbb{N} \cup \{0\}$. The projection onto the k-th Landau level is denoted by $\Pi_{B,k}$. The spectral theorem implies that the operator $\varphi_{\lambda,\gamma}(H_B)$ can then be written as

$$\varphi_{\lambda,\gamma}(H_B) = \sum_{k=0}^\infty \varphi_{\lambda,\gamma}\left((2k+1)B\right) \Pi_{B,k}.$$

We multiply this identity from both sides with the projection $P_{\Omega^{\mathrm{c}}}$ and consider the trace of the obtained expression, that is

$$\operatorname{tr}\left(P_{\Omega^{\mathrm{c}}}\varphi_{\lambda,\gamma}(H_B)P_{\Omega^{\mathrm{c}}}\right) = \sum_{k=0}^\infty \varphi_{\lambda,\gamma}\left((2k+1)B\right) \operatorname{tr}(P_{\Omega^{\mathrm{c}}}\Pi_{B,k}). \tag{13}$$

To explicitly calculate the summands on the right-hand-side of (13), we observe that by the cyclicity of the trace

$$\operatorname{tr}(P_{\Omega^{\mathrm{c}}}\Pi_{B,k}) = \operatorname{tr}(P_{\Omega^{\mathrm{c}}}\Pi_{B,k}\Pi_{B,k}P_{\Omega^{\mathrm{c}}}) = \left\| P_{\Omega^{\mathrm{c}}}\Pi_{B,k} \right\|_{\sigma_2}^2, \tag{14}$$

where $\|\cdot\|_{\sigma_2}$ denotes the Hilbert–Schmidt norm. This norm can be calculated explicitly by using the integral kernel of the operator $P_{\Omega^{\mathrm{c}}}\Pi_{B,k}$. Let $\Pi_{B,k}(x,y)$ be the integral kernel of $\Pi_{B,k}$ such that $\Pi_{B,k}\psi(x) = \int_{\mathbb{R}^2} \Pi_{B,k}(x,y)\psi(y)\,\mathrm{d}y$. The integral kernel of the composition $P_{\Omega^{\mathrm{c}}}\Pi_{B,k}$ is then given by

$$(P_{\Omega^{\mathrm{c}}}\Pi_{B,k})(x,y) = \chi_{\Omega^{\mathrm{c}}}(x)\Pi_{B,k}(x,y).$$

We can calculate the Hilbert-Schmidt norm on the right-hand-side of (14) by double integration of the square of the modulus of this integral kernel, that is

$$\operatorname{tr}(P_{\Omega^c}\Pi_{B,k}) = \left\| P_{\Omega^c}\Pi_{B,k} \right\|_{\sigma_2}^2 = \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} |\Pi_{B,k}(x,y)|^2 \chi_{\Omega^c}(x)\,\mathrm{d}y\,\mathrm{d}x. \tag{15}$$

To explicitly solve this integral, we point out some important properties of the function $\Pi_{B,k}(x,y)$. As the orthogonal projection $\Pi_{B,k}$ is self-adjoint it must hold that $\Pi_{B,k}(x,y) = \overline{\Pi_{B,k}(y,x)}$. By evaluating $\Pi_{B,k}$ at the delta distribution $\delta(x-x_0)$ and using the defining property of projections, $\Pi_{B,k} = \Pi_{B,k}\Pi_{B,k}$, it can easily be concluded that

$$\int\limits_{\mathbb{R}^2} |\Pi_{B,k}(x_0,y)|^2\,\mathrm{d}y = \Pi_{B,k}(x_0,x_0). \tag{16}$$

It is furthermore a remarkable fact that the diagonal of the integral kernel of $\Pi_{B,k}$ is given by the constant $\Pi_{B,k}(x,x) = \frac{B}{2\pi}$ for all $k \in \mathbb{N}\cup\{0\}$. Using these properties, identity (15) can be continued as

$$\operatorname{tr}(P_{\Omega^c}\Pi_{B,k}) = \int\limits_{\mathbb{R}^2} \Pi_{B,k}(x,x)\chi_{\Omega^c}(x)\,\mathrm{d}x = \frac{B}{2\pi}|\mathbb{R}^2 \setminus \Omega|.$$

Inserting this equation back into (13) yields the final result

$$\operatorname{tr}\left(P_{\Omega^c}\varphi_{\lambda,\gamma}(H_B)P_{\Omega^c}\right) = |\mathbb{R}^2\setminus\Omega|\frac{B}{2\pi}\sum_{k=0}^{\infty}\varphi_{\lambda,\gamma}\left((2k+1)B\right)$$

which finishes the proof.

4 The proof of Theorem 3

Let the convex function $\varphi_{\lambda,\gamma}:\mathbb{R}\to\mathbb{R}$ be defined as in (12). Theorem 1 applied to L yields (10) and it only remains to show (11) which can be proven following calculations by Hansson and Laptev [8]. Firstly, we introduce the Fourier transformation $\mathscr{F}_{x_3}$ with respect to the variable x_3,

$$\mathscr{F}_{x_3}\psi(x_1,x_2,x_3) = (2\pi)^{-\frac{1}{2}}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x_3y_3}\psi(x_1,x_2,y_3)\,\mathrm{d}y_3.$$

A simple calculation shows that the Heisenberg sub-Laplacian L satisfies the identity

$$\mathscr{F}_{x_3}L\mathscr{F}_{x_3}^* = \left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{1}{2}x_2x_3\right)^2 + \left(\mathrm{i}\frac{\partial}{\partial x_2} + \frac{1}{2}x_1x_3\right)^2. \tag{17}$$

For fixed x_3, the right-hand-side of (17) can be identified with a two-dimensional Schrödinger operator with vector field $A = \frac{x_3}{2}(-x_2, x_1)$. The corresponding magnetic field $B = \mathrm{d}A$ is constant and can be calculated to be $B = x_3$. The eigenvalues of this magnetic operator H_{x_3} are $(2k+1)|x_3|$ for $k \in \mathbb{N} \cup \{0\}$. Note that they depend on the variable x_3. Similar to the previous section we shall use the Landau projections $\Pi_{x_3,k}$ to prove (11). With respect to the variable x_3, the operator $\mathscr{F}_{x_3} L \mathscr{F}_{x_3}^*$ simply acts as a multiplication operator and consequently the spectral theorem allows us to write

$$\varphi_{\lambda,\gamma}(\mathscr{F}_{x_3} L \mathscr{F}_{x_3}^*) = \sum_{k=0}^{\infty} \varphi_{\lambda,\gamma}\left((2k+1)|x_3|\right) \widehat{\Pi}_{x_3,k} \tag{18}$$

where we have used the tensor product $\widehat{\Pi}_{x_3,k} = \Pi_{x_3,k} \otimes \mathbb{I}_{L^2(\mathbb{R})}$ with $\mathbb{I}_{L^2(\mathbb{R})}$ denoting the identity on $L^2(\mathbb{R})$. For convenience we introduce the notation $\mu_k(x_3) = \varphi_{\lambda,\gamma}\left((2k+1)|x_3|\right)$ swallowing the dependence on λ and γ for the moment. As a consequence of (18) we obtain the identity

$$\operatorname{tr}\left(P_{\Omega^c} \varphi_{\lambda,\gamma}(L) P_{\Omega^c}\right) = \sum_{k=0}^{\infty} \operatorname{tr}\left(P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3) \widehat{\Pi}_{x_3,k} \mathscr{F}_{x_3} P_{\Omega^c}\right). \tag{19}$$

This result can be compared to the analogous equation in the case of a magnetic operator (13). While in this setting the magnetic field was a given constant, we now consider a magnetic field that changes with the variable x_3. In addition, the Fourier transformation $\mathscr{F}_{x_3}$ has to be dealt with. The summands on the right-hand-side of (19) can be written as Hilbert–Schmidt norms of certain operators, that is

$$\operatorname{tr}\left(P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3) \widehat{\Pi}_{x_3,k} \mathscr{F}_{x_3} P_{\Omega^c}\right) = \left\| P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3)^{\frac{1}{2}} \widehat{\Pi}_{x_3,k} \right\|_{\sigma_2}^2$$

for every $k \in \mathbb{N} \cup \{0\}$. Here we have used that the multiplication operator $\mu_k(x_3)$ and the projection $\Pi_{x_3,k}$ commute. The investigation of these Hilbert–Schmidt norms requires us to calculate the integral kernels of the operators involved. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The integral kernel of $P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3)^{\frac{1}{2}} \widehat{\Pi}_{x_3,k}$ can then be computed to be $\chi_{\Omega^c}(x) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i} x_3 y_3} \Pi_{y_3,k}(x_1, x_2, y_1, y_2) \mu_k(y_3)^{\frac{1}{2}}$. As a consequence we obtain the identity

$$\begin{aligned}
&\left\| P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3)^{\frac{1}{2}} \widehat{\Pi}_{x_3,k} \right\|_{\sigma_2}^2 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_{\Omega^c}(x) |\Pi_{y_3,k}(x_1, x_2, y_1, y_2)|^2 \mu_k(y_3) \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mu_k(y_3) \int_{\mathbb{R}^3} \chi_{\Omega^c}(x) \int_{\mathbb{R}^2} |\Pi_{y_3,k}(x_1, x_2, y_1, y_2)|^2 \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}x \, \mathrm{d}y_3.
\end{aligned}$$

To calculate this integral, we recall (16) and stress again that the diagonal of the integral kernel $\Pi_{y_3,k}(x_1, x_2, y_1, y_2)$ is is known to be the constant $\frac{|y_3|}{2\pi}$. This results in

$$\left\| P_{\Omega^c} \mathscr{F}_{x_3}^* \mu_k(x_3)^{\frac{1}{2}} \widehat{\Pi}_{x_3,k} \right\|_{\sigma_2}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \mu_k(y_3) \int_{\mathbb{R}^3} \chi_{\Omega^c}(x) \frac{|y_3|}{2\pi} \, \mathrm{d}x \, \mathrm{d}y_3$$

$$= \frac{1}{2\pi^2} |\mathbb{R}^3 \setminus \Omega| \int_0^{+\infty} ((2k+1)y_3 - \lambda)_-^\gamma \, y_3 \, \mathrm{d}y_3$$

where we have used the definition of $\mu_k(y_3)$ to obtain the last equality. We insert this identity back into (19) and substitute $p = (2k+1)y_3$ to conclude that

$$\operatorname{tr}\left(P_{\Omega^c} \varphi_{\lambda,\gamma}(L) P_{\Omega^c} \right) = \frac{1}{2\pi^2} |\mathbb{R}^3 \setminus \Omega| \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \int_0^{+\infty} (p - \lambda)_-^\gamma \, p \, \mathrm{d}p$$

$$= \frac{1}{16} |\mathbb{R}^3 \setminus \Omega| \int_0^{+\infty} (p - \lambda)_-^\gamma \, p \, \mathrm{d}p.$$

Here, the last equality follows from the well-known fact that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$. The remaining integral can be easily calculated using partial integration

$$\int_0^{+\infty} (p - \lambda)_-^\gamma \, , p \, \mathrm{d}p = \frac{\lambda^{\gamma+2}}{(\gamma+1)(\gamma+2)}$$

and this yields the desired result.

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